Discrete Probability Distributions

Chapter 6

McGraw-Hill/Irwin

GOALS

1. Define the terms *probability distribution* and *random variable*.
2. Distinguish between *discrete* and *continuous probability distributions*.
3. Calculate the *mean, variance*, and *standard deviation* of a discrete probability distribution.
4. Describe the characteristics of and compute probabilities using the *binomial probability distribution*.
5. Describe the characteristics of and compute probabilities using the *hypergeometric probability distribution*.
6. Describe the characteristics of and compute probabilities using the *Poisson probability distribution*.

What is a Probability Distribution?

PROBABILITY DISTRIBUTION A listing of all the outcomes of an experiment and the probability associated with each outcome.

Experiment:

Toss a coin three times. Observe the number of heads. The possible results are: Zero heads, One head, Two heads, and Three heads.

What is the probability distribution for the number of heads?

Possible Result	Coin Toss First	Second	Third	Number of Heads
1	T	T	T	0
2	T	T	H	1
3	T	H	T	1
4	T	H	H	2
5	H	T	T	1
6	H	T	H	2
7	H	H	T	2
8	H	H	H	3

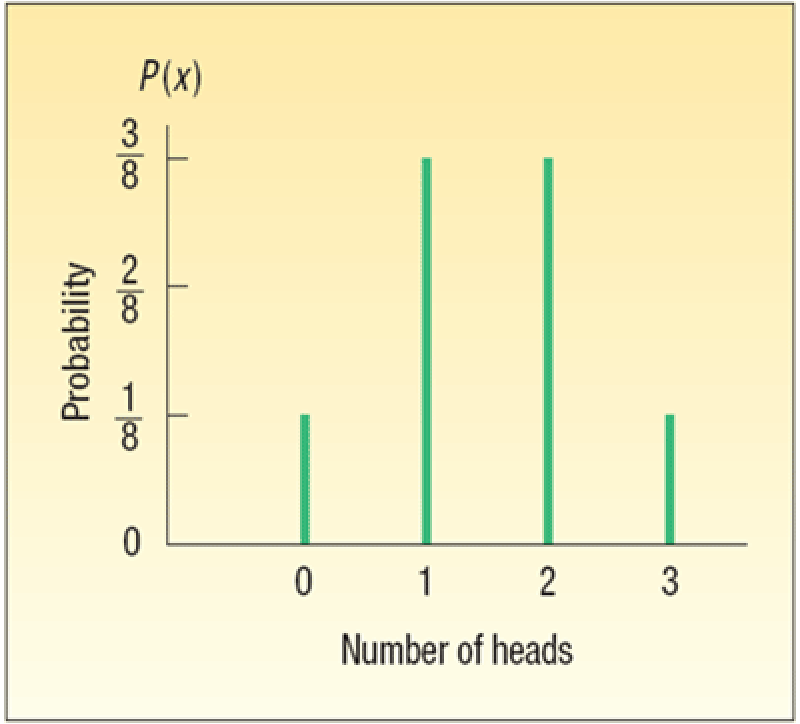
Characteristics of a Probability Distribution

CHARACTERISTICS OF A PROBABILITY DISTRIBUTION

1. The probability of a particular outcome is between 0 and 1 inclusive.
2. The outcomes are mutually exclusive events.
3. The list is exhaustive. So the sum of the probabilities of the various events is equal to 1.

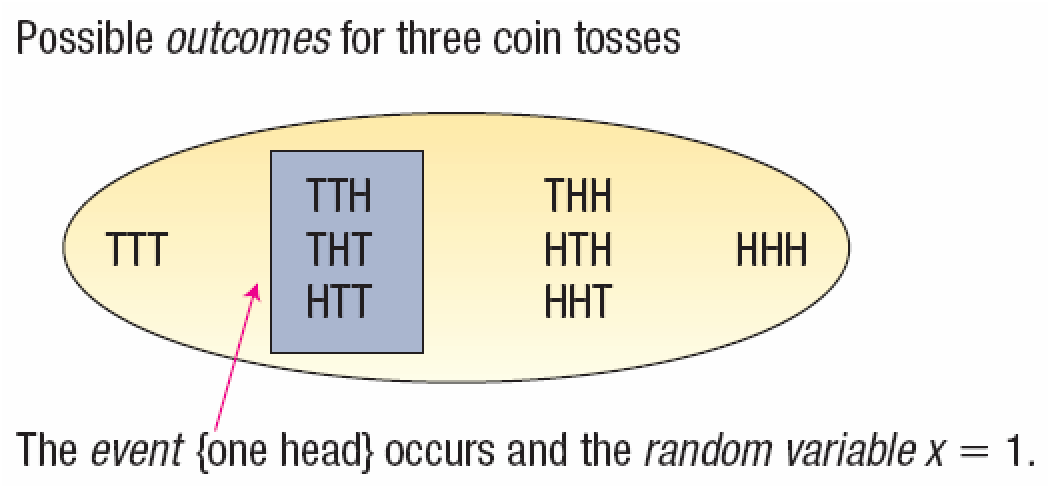
Probability Distribution of Number of Heads Observed in 3 Tosses of a Coin

Number of Heads, x	Probability of Outcome, $P(x)$
0	$\frac{1}{8} = .125$
1	$\frac{3}{8} = .375$
2	$\frac{3}{8} = .375$
3	$\frac{1}{8} = .125$
Total	$\frac{8}{8} = 1.000$



Random Variables

RANDOM VARIABLE A quantity resulting from an experiment that, by chance, can assume different values.



Types of Random Variables

DISCRETE RANDOM VARIABLE A random variable that can assume only certain clearly separated values. It is usually the result of counting something.

CONTINUOUS RANDOM VARIABLE can assume an infinite number of values within a given range. It is usually the result of some type of <u>measurement</u>

Discrete Random Variables

DISCRETE RANDOM VARIABLE A random variable that can assume only certain clearly separated values. It is usually the result of counting something.

EXAMPLES

1. The number of students in a class.
2. The number of children in a family.
3. The number of cars entering a carwash in a hour.
4. Number of home mortgages approved by Coastal Federal Bank last week.

Continuous Random Variables

CONTINUOUS RANDOM VARIABLE can assume an infinite number of values within a given range. It is usually the result of some type of measurement

EXAMPLES

- The length of each song on the latest Tim McGraw album.
- The weight of each student in this class.
- The temperature outside as you are reading this book.
- The amount of money earned by each of the more than 750 players currently on Major League Baseball team rosters.

The Mean of a Probability Distribution

MEAN

•The mean is a typical value used to represent the central location of a probability distribution.

•The mean of a probability distribution is also referred to as its **expected value.**

MEAN OF A PROBABILITY DISTRIBUTION $\mu = \Sigma[xP(x)]$ [6–1]

The Variance and Standard Deviation of a Probability Distribution

VARIANCE AND STANDARD DEVIATION

- Measures the amount of spread in a distribution
- The computational steps are:

  1. Subtract the mean from each value, and square this difference.
  2. Multiply each squared difference by its probability.
  3. Sum the resulting products to arrive at the variance.

  The standard deviation is found by taking the positive square root of the variance.

VARIANCE OF A PROBABILITY DISTRIBUTION $\sigma^2 = \Sigma[(x - \mu)^2 P(x)]$ **[6–2]**

Mean, Variance, and Standard Deviation of a Probability Distribution - Example

John Ragsdale sells new cars for Pelican Ford. John usually sells the largest number of cars on Saturday. He has developed the following probability distribution for the number of cars he expects to sell on a particular Saturday.

Number of Cars Sold, x	Probability, $P(x)$
0	.10
1	.20
2	.30
3	.30
4	.10
Total	1.00

Mean of a Probability Distribution - Example

$$\mu = \Sigma[xP(x)]$$
$$= 0(.10) + 1(.20) + 2(.30) + 3(.30) + 4(.10)$$
$$= 2.1$$

Number of Cars Sold, x	Probability, $P(x)$	$x \cdot P(x)$
0	.10	0.00
1	.20	0.20
2	.30	0.60
3	.30	0.90
4	.10	0.40
Total	1.00	$\mu = 2.10$

Variance and Standard Deviation of a Probability Distribution - Example

$$\sigma^2 = \Sigma[(x - \mu)^2 P(x)]$$

Number of Cars Sold, x	Probability, $P(x)$	$(x - \mu)$	$(x - \mu)^2$	$(x - \mu)^2 P(x)$
0	.10	0 − 2.1	4.41	0.441
1	.20	1 − 2.1	1.21	0.242
2	.30	2 − 2.1	0.01	0.003
3	.30	3 − 2.1	0.81	0.243
4	.10	4 − 2.1	3.61	0.361
				$\sigma^2 = 1.290$

$$\sigma = \sqrt{\sigma^2} = \sqrt{1.290} = 1.136$$

Binomial Probability Distribution

A Widely occurring discrete probability distribution

Characteristics of a Binomial Probability Distribution

1. There are only two possible outcomes on a particular trial of an experiment.
2. The outcomes are mutually exclusive,
3. The random variable is the result of counts.
4. Each trial is *independent* of any other trial

Binomial Probability Experiment

1. An outcome on each trial of an experiment is classified into one of two mutually exclusive categories—a *success or a failure*.
2. The random variable counts the number of successes in a *fixed number of trials*.
3. The *probability of success and failure stay the same* for each trial.
4. The *trials are independent*, meaning that the outcome of one trial does not affect the outcome of any other trial.

Binomial Probability Formula

BINOMIAL PROBABILITY FORMULA	$P(x) = {}_nC_x \, \pi^x(1 - \pi)^{n - x}$	**[6–3]**

where:

C denotes a combination.
n is the number of trials.
x is the random variable defined as the number of successes.
π is the probability of a success on each trial.

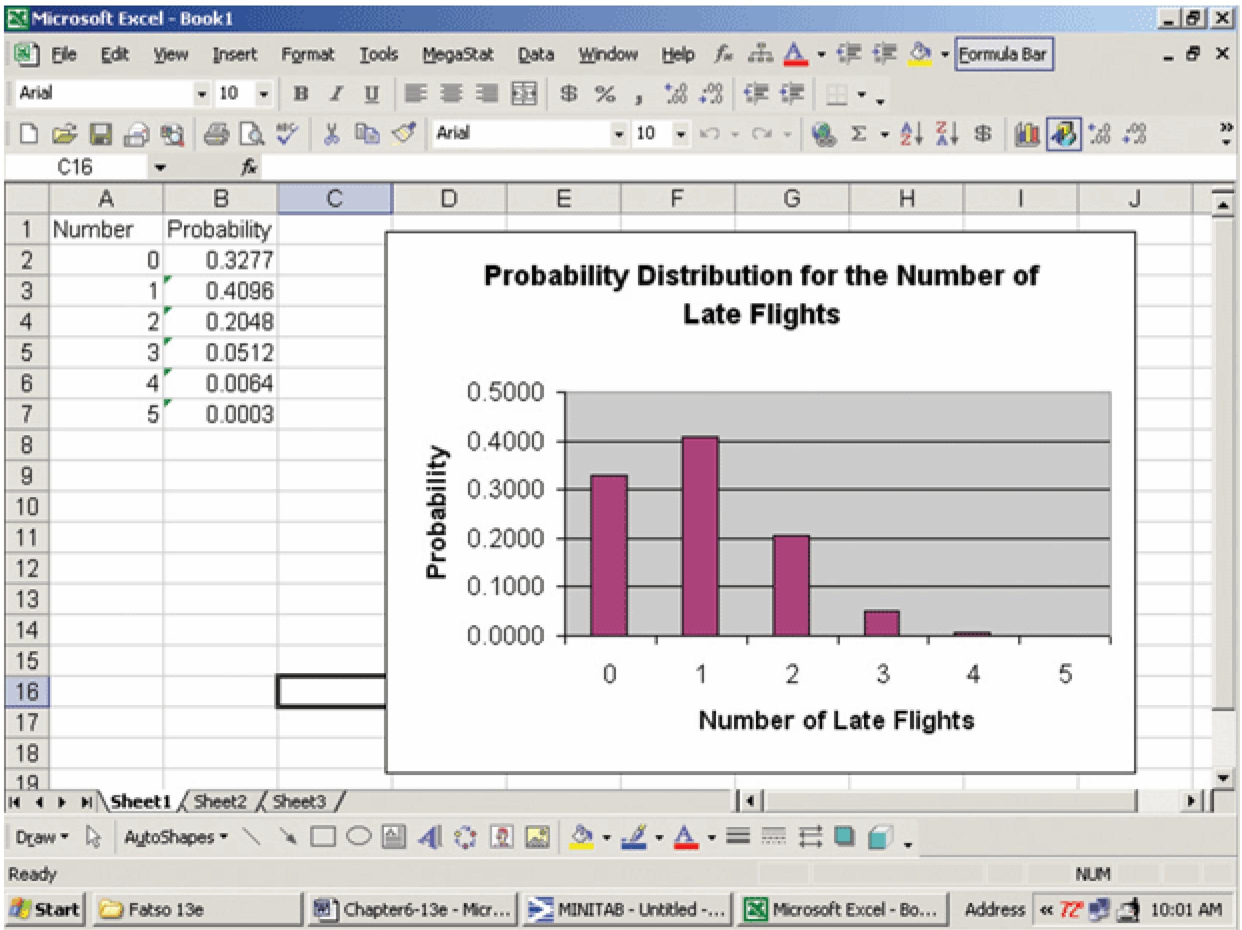
Binomial Probability - Example

There are five flights daily from Pittsburgh via US Airways into the Bradford, Pennsylvania, Regional Airport. Suppose the probability that any flight arrives late is .20.

What is the probability that none of the flights are late today?

$$
\begin{aligned}
P(x=0) &= {}_nC_x \pi^x (1-\pi)^{n-x} \\
&= {}_5C_0 (.20)^0 (1-.20)^{5-0} \\
&= (1)(1)(.3277) \\
&= 0.3277
\end{aligned}
$$

Binomial Probability - Excel



Binomial Dist. – Mean and Variance

MEAN OF A BINOMIAL DISTRIBUTION $$\mu = n\pi \qquad [6\text{–}4]$$

VARIANCE OF A BINOMIAL DISTRIBUTION $$\sigma^2 = n\pi(1 - \pi) \qquad [6\text{–}5]$$

Binomial Dist. – Mean and Variance: Example

For the example regarding the number of late flights, recall that $\pi = .20$ and $n = 5$.

What is the average number of late flights?

What is the variance of the number of late flights?

$$\mu = n\pi$$
$$= (5)(0.20) = 1.0$$

$$\sigma^2 = n\pi(1-\pi)$$
$$= (5)(0.20)(1-0.20)$$
$$= (5)(0.20)(0.80)$$
$$= 0.80$$

Binomial Dist. – Mean and Variance: Another Solution

Number of Late Flights, x	$P(x)$	$xP(x)$	$x - \mu$	$(x - \mu)^2$	$(x - \mu)^2 P(x)$
0	0.3277	0.0000	−1	1	0.3277
1	0.4096	0.4096	0	0	0
2	0.2048	0.4096	1	1	0.2048
3	0.0512	0.1536	2	4	0.2048
4	0.0064	0.0256	3	9	0.0576
5	0.0003	0.0015	4	16	0.0048
		$\mu = 1.0000$			$\sigma^2 = 0.7997$

$$\mu = \Sigma[xP(x)]$$

$$\sigma^2 = \Sigma[(x - \mu)^2 P(x)]$$

Binomial Distribution - Table

Five percent of the worm gears produced by an automatic, high-speed Carter-Bell milling machine are defective.

What is the probability that out of six gears selected at random none will be defective? Exactly one? Exactly two? Exactly three? Exactly four? Exactly five? Exactly six out of six?

TABLE 6–2 Binomial Probabilities for $n = 6$ and Selecte Values of π

$n = 6$ Probability											
$x\backslash\pi$	.05	.1	.2	.3	.4	.5	.6	.7	.8	.9	.95
0	.735	.531	.262	.118	.047	.016	.004	.001	.000	.000	.000
1	.232	.354	.393	.303	.187	.094	.037	.010	.002	.000	.000
2	.031	.098	.246	.324	.311	.234	.138	.060	.015	.001	.000
3	.002	.015	.082	.185	.276	.313	.276	.185	.082	.015	.002
4	.000	.001	.015	.060	.138	.234	.311	.324	.246	.098	.031
5	.000	.000	.002	.010	.037	.094	.187	.303	.393	.354	.232
6	.000	.000	.000	.001	.004	.016	.047	.118	.262	.531	.735

Binomial Distribution - MegaStat

Five percent of the worm gears produced by an automatic, high-speed Carter-Bell milling machine are defective.

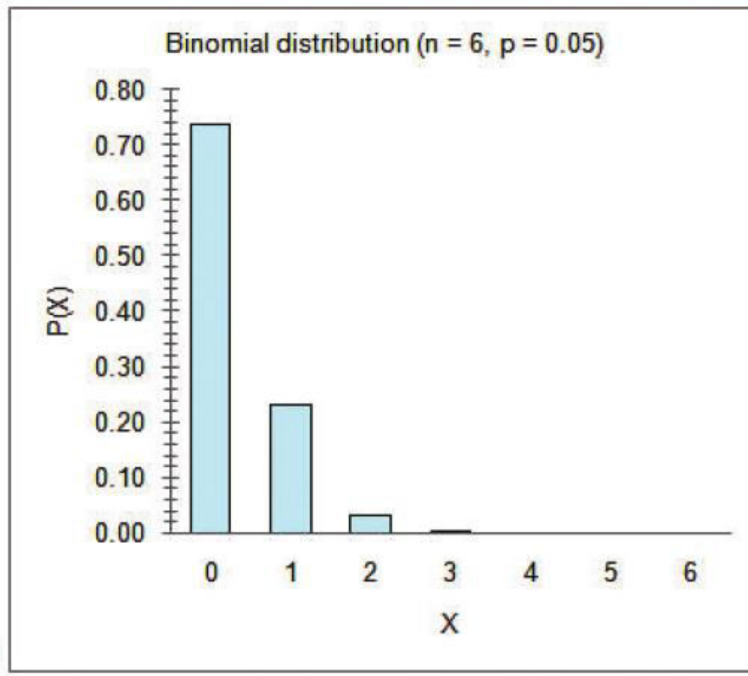
What is the probability that out of six gears selected at random none will be defective? Exactly one? Exactly two? Exactly three? Exactly four? Exactly five? Exactly six out of six?

Binomial distribution

6	n
0.05	p

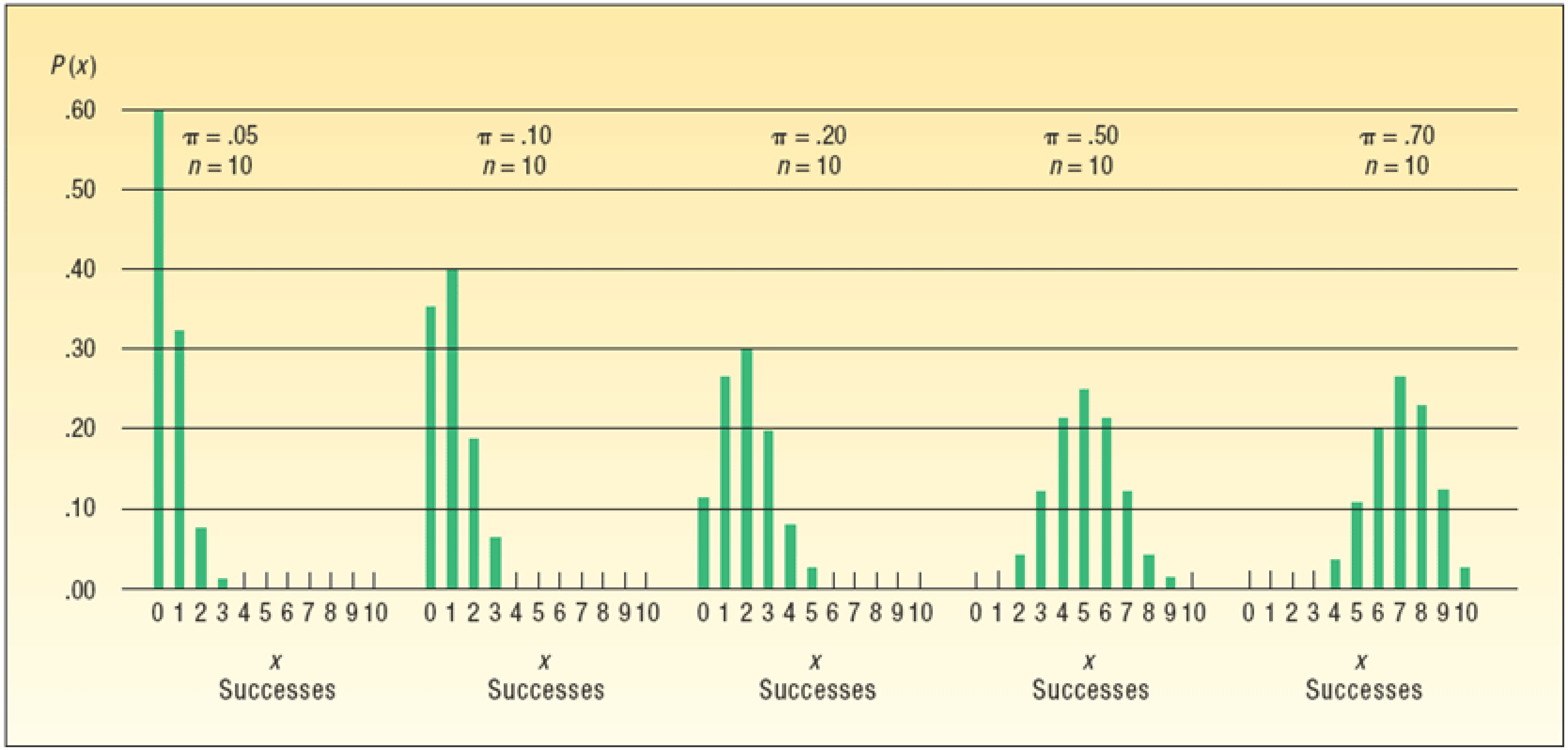
X	P(X)	cumulative probability
0	0.73509	0.73509
1	0.23213	0.96723
2	0.03054	0.99777
3	0.00214	0.99991
4	0.00008	1.00000
5	0.00000	1.00000
6	0.00000	1.00000
	1.00000	

0.300	expected value
0.285	variance
0.534	standard deviation

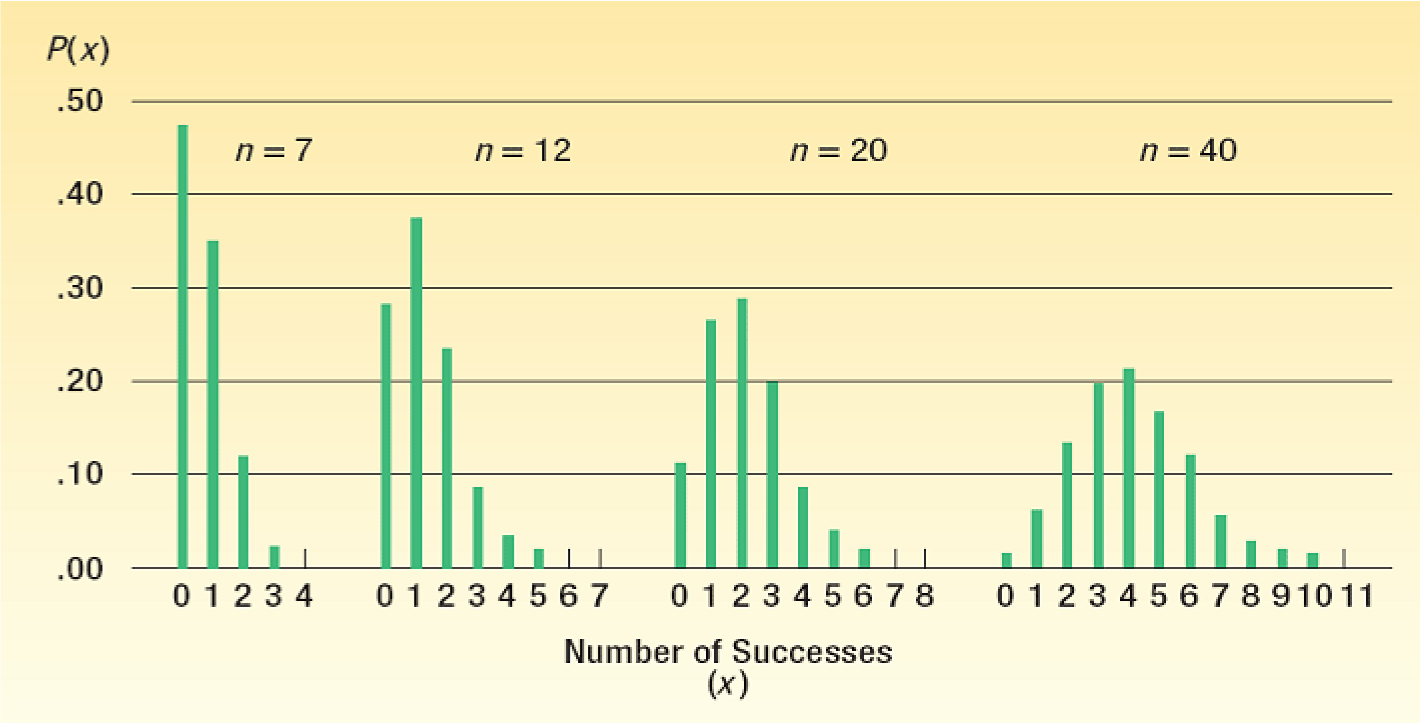


Binomial – Shapes for Varying π (*n* constant)

CHART 6–2 Graphing the Binomial Probability Distribution for a π of .05, .10, .20, .50, and .70 and an *n* of 10



Binomial – Shapes for Varying *n* (π constant)



Binomial Probability Distributions - Example

A study by the Illinois Department of Transportation concluded that 76.2 percent of front seat occupants used seat belts. A sample of 12 vehicles is selected.

What is the probability the front seat occupants in exactly 7 of the 12 vehicles are wearing seat belts?

$$
\begin{aligned}
P(x = 7 \mid n = 12 \text{ and } \pi = .762) &= {}_{12}C_7(.762)^7(1 - .762)^{12-7} \\
&= 792(.149171)(.000764) = .0902
\end{aligned}
$$

Cumulative Binomial Probability Distributions - Example

A study by the Illinois Department of Transportation concluded that 76.2 percent of front seat occupants used seat belts. A sample of 12 vehicles is selected.

What is the probability the front seat occupants in at least 7 of the 12 vehicles are wearing seat belts?

$$
\begin{aligned}
P(x \geq 7 \mid n = 12 \text{ and } \pi = .762 \\
&= P(x = 7) + P(x = 8) + P(x = 9) + P(x = 10) + P(x = 11) + P(x = 12) \\
&= .0902 + .1805 + .2569 + .2467 + .1436 + .0383 \\
&= .9562
\end{aligned}
$$

Cumulative Binomial Probability Distributions - Excel

	A	B
1	Wearing Seat Belts	Probability
2	0	
3	1	
4	2	
5	3	
6	4	
7	5	
8	6	
9	7	0.0902
10	8	0.1805
11	9	0.2569
12	10	0.2467
13	11	0.1436
14	12	0.0383
15		0.9563
16		
17		

Probability

Hypergeometric Probability Distribution

1. An outcome on each trial of an experiment is classified into one of two mutually exclusive categories—a *success or a failure*.
2. The *probability of success and failure changes* from trial to trial.
3. The *trials are not independent*, meaning that the outcome of one trial affects the outcome of any other trial.

Note: Use hypergeometric distribution if experiment is binomial, but sampling is without replacement from a finite population where n/N is more than 0.05

Hypergeometric Probability Distribution - Formula

HYPERGEOMETRIC DISTRIBUTION $$P(x) = \frac{({}_SC_x)({}_{N-S}C_{n-x})}{{}_NC_n} \qquad [6\text{–}6]$$

where:

- N is the size of the population.
- S is the number of successes in the population.
- x is the number of successes in the sample. It may be 0, 1, 2, 3, ...
- n is the size of the sample or the number of trials.
- C is the symbol for a combination.

Hypergeometric Probability Distribution - Example

PlayTime Toys, Inc., employs **50** people in the Assembly Department. **Forty** of the employees belong to a union and **ten** do not. **Five** employees are selected at random to form a committee to meet with management regarding shift starting times.

What is the probability that four of the five selected for the committee belong to a union?

Hypergeometric Probability Distribution - Example

Here's what's given:

N = 50 (number of employees)

S = 40 (number of union employees)

x = 4 (number of union employees selected)

n = 5 (number of employees selected)

What is the probability that **four** of the five selected for the committee belong to a union?

$$P(x) = \frac{({}_SC_x)({}_{N-S}C_{n-x})}{{}_NC_n}$$

$$P(4) = \frac{({}_{40}C_4)({}_{50-40}C_{5-4})}{{}_{50}C_5} = \frac{\left(\frac{40!}{4!36!}\right)\left(\frac{10!}{1!9!}\right)}{\frac{50!}{5!45!}} = \frac{(91{,}390)(10)}{2{,}118{,}760} = .431$$

Poisson Probability Distribution

The **Poisson probability distribution** describes the number of times some event occurs during a specified interval. The interval may be time, distance, area, or volume.

Assumptions of the Poisson Distribution

(1) The probability is proportional to the length of the interval.

(2) The intervals are independent.

Poisson Probability Distribution

The Poisson probability distribution is characterized by the number of times an event happens during some interval or continuum.

Examples include:

- The number of misspelled words per page in a newspaper.
- The number of calls per hour received by Dyson Vacuum Cleaner Company.
- The number of vehicles sold per day at Hyatt Buick GMC in Durham, North Carolina.
- The number of goals scored in a college soccer game.

Poisson Probability Distribution

The Poisson distribution can be described mathematically using the formula:

POISSON DISTRIBUTION $$P(x) = \frac{\mu^x e^{-\mu}}{x!} \quad [6\text{–}7]$$

where:

μ (mu) is the mean number of occurrences (successes) in a particular interval.
e is the constant 2.71828 (base of the Napierian logarithmic system).
x is the number of occurrences (successes).
$P(x)$ is the probability for a specified value of x.

Poisson Probability Distribution

- The mean number of successes μ can be determined in binomial situations by $n\pi$, where n is the number of trials and π the probability of a success.

MEAN OF A POISSON DISTRIBUTION $$\mu = n\pi \qquad [6\text{–}8]$$

- The variance of the Poisson distribution is also equal to $n\pi$.

Poisson Probability Distribution - Example

Assume baggage is rarely lost by Northwest Airlines. Suppose a random sample of 1,000 flights shows a total of 300 bags were lost. Thus, the arithmetic mean number of lost bags per flight is 0.3 (300/1,000). If the number of lost bags per flight follows a Poisson distribution with $u = 0.3$, find the probability of not losing any bags.

$$P(0) = \frac{\mu^{x} e^{-u}}{x!} = \frac{0.3^{0} e^{-.3}}{0!} = .7408$$

Poisson Probability Distribution - Table

Recall from the previous illustration that the number of lost bags follows a Poisson distribution with a mean of 0.3. Use Appendix B.5 to find the probability that no bags will be lost on a particular flight. What is the probability exactly one bag will be lost on a particular flight?

TABLE 6–6 Poisson Table for Various Values of μ (from Appendix B.5)

	μ								
x	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066
1	0.0905	0.1637	0.2222	0.2681	0.3033	0.3293	0.3476	0.3595	0.3659
2	0.0045	0.0164	0.0333	0.0536	0.0758	0.0988	0.1217	0.1438	0.1647
3	0.0002	0.0011	0.0033	0.0072	0.0126	0.0198	0.0284	0.0383	0.0494
4	0.0000	0.0001	0.0003	0.0007	0.0016	0.0030	0.0050	0.0077	0.0111
5	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004	0.0007	0.0012	0.0020
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003
7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

More About the Poisson Probability Distribution

- The Poisson probability distribution is always positively skewed and the random variable has no specific upper limit.
- The Poisson distribution for the lost bags illustration, where μ=0.3, is highly skewed. As μ becomes larger, the Poisson distribution becomes more symmetrical.