Calculus 1

Lecture 11:

Integral

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Anti-derivatives

DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as F to represent an antiderivative of a function f, G to represent an antiderivative of g, and so forth.

EXAMPLE 1 Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$ **(b)** $g(x) = \cos x$

General form of anti-derivatives

THEOREM 8 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Example 1:

Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Answer:

$$F(x) = x^3 + c$$

$F(1) = -1 \Rightarrow F(1) = (1)^3 + c = -1 \Rightarrow c = -2$

$$F(x) = x^3 - 2$$

Anti-derivative formulas

Function	General antiderivative	Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	**8.** e^{kx}	$\frac{1}{k}e^{kx} + C$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$	**9.** $\frac{1}{x}$	$\ln\lvert x\rvert + C, \quad x \neq 0$
3. cos kx	$\frac{1}{k}\sin kx + C$	**10.** $\frac{1}{\sqrt{1 - k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$	**11.** $\frac{1}{1 + k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	**12.** $\frac{1}{x\sqrt{k^2x^2 - 1}}$	$\sec^{-1} kx + C, kx > 1$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	**13.** a^{kx}	$\left(\frac{1}{k \ln a}\right)a^{kx} + C, a > 0, a \neq 1$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$		

k a nonzero constant

Anti-derivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule*:	$kf(x)$	$kF(x) + C$, $\quad k$ a constant
2.	*Negative Rule*:	$-f(x)$	$-F(x) + C$
3.	*Sum or Difference Rule*:	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

Indefinite integral

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$F(x) = \int f(x)\,dx$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Example 2:

Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$

Evaluate

$$\int (x^2 - 2x + 5)\, dx.$$

Answer:

$$F(x) = \int (x^2 - 2x + 5)\, dx$$

$$= \frac{1}{3}x^3 - x^2 + 5x + c$$

Example 3:

Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$
3. cos kx	$\frac{1}{k}\sin kx + C$

Evaluate

$$y = \int \sin 3x \; dx$$

Answer:

$$y = -\frac{1}{3}\cos 3x + c$$

Example 4:

Evaluate

$$y = \int \cos 5x \ dx$$

Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$
3. cos kx	$\frac{1}{k}\sin kx + C$

Answer:

$$y = \frac{1}{5}\sin 5x + c$$

Example 5:

Function	General antiderivative
8. e^{kx}	$\frac{1}{k}e^{kx} + C$

Evaluate

$$y = \int e^{2x}\ dx$$

Answer:

$$y = \frac{1}{2}e^{2x} + c$$

Example 6:

Evaluate

$$y = \int 3x^2 \; dx \qquad , \qquad y(2) = 10$$

Answer:

$$y = x^3 + c$$

$$y(2) = 2^3 + c$$

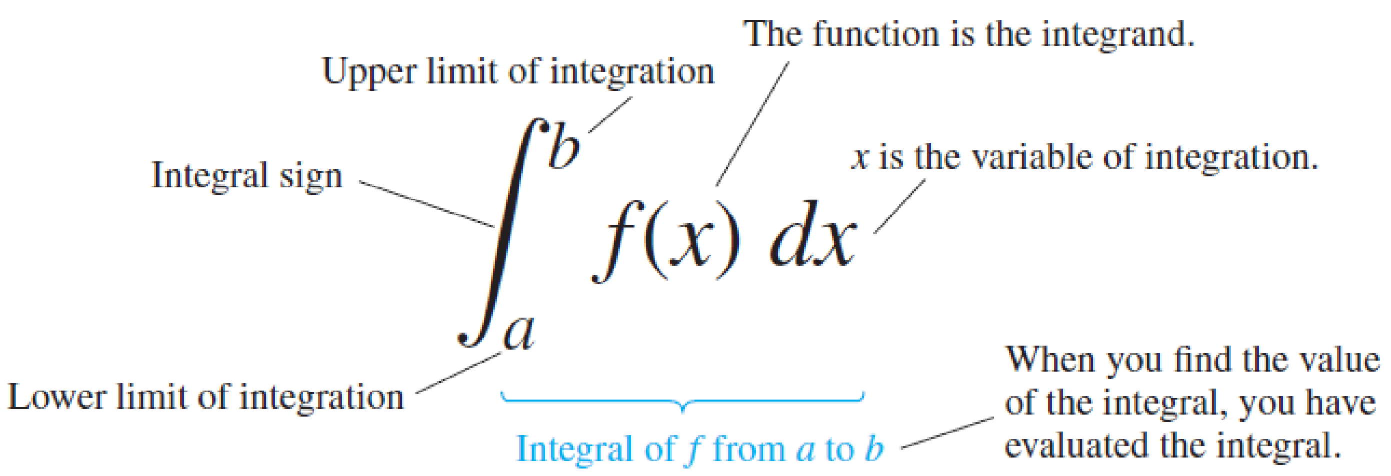
$$10 = 8 + c \;\; \Rightarrow \;\; c = 2 \qquad \Rightarrow \qquad y = x^3 + 2$$

Definite integral

The symbol for the number J in the definition of the **definite integral** is

$$\int_a^b f(x)\,dx,$$

which is read as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x." The component parts in the integral symbol also have names:



Example 7:

Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$

Evaluate

$$y = \int_0^2 2x\, dx$$

Answer:

$$y = \int_0^2 2x\, dx$$

$$= x^2 \Big|_0^2$$

$$= 2^2 - 0^2 = 4 - 0 = 4$$

Example 7:

Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$

Evaluate

$$y = \int_0^{\pi/4} \cos 2x \; dx$$

Answer:

$$y = \int_0^{\pi/4} \cos 2x \; dx$$

$$= \frac{1}{2} \sin 2x \Big|_0^{\pi/4}$$

$$= \frac{1}{2}\left[\sin\left(2 \times \frac{\pi}{4}\right) - \sin(2 \times 0)\right]$$

$$= \frac{1}{2}\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] = \frac{1}{2}[1 - 0] = \frac{1}{2}$$

Properties of definite integral

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A definition when $f(a)$ exists

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Properties of definite integral (cont'd)

6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b - a) \leq \int_a^b f(x)\,dx \leq \max f \cdot (b - a).$$

7. *Domination:*

$$f(x) \geq g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

$$f(x) \geq 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \geq 0$$ (Special case)

Area under the graph of a nonnegative function

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$