Calculus 1

Lecture 3:

Limit

By:

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Overview

- Mathematicians of the seventeenth century were keenly interested in the study of motion for objects on or near the earth and the motion of planets and stars.

- This study involved both the speed of the object and its direction of motion at any instant, and they knew the direction at a given instant was along a line tangent to the path of motion.

- The concept of a limit is fundamental to finding the velocity of a moving object and the tangent to a curve.

Average rate of changes and secant lines

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

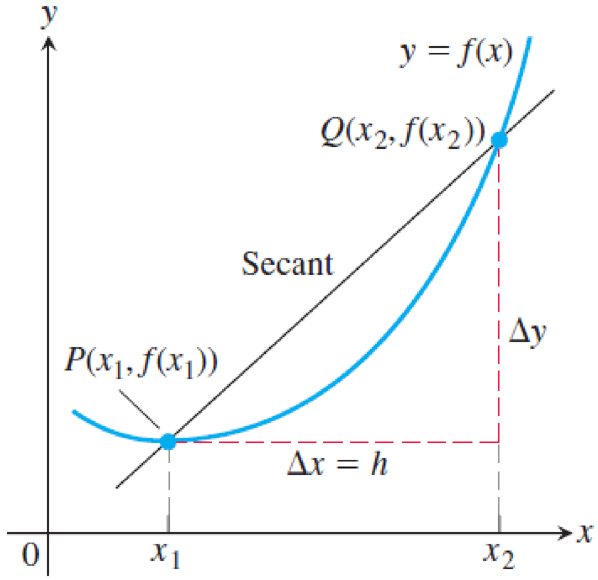
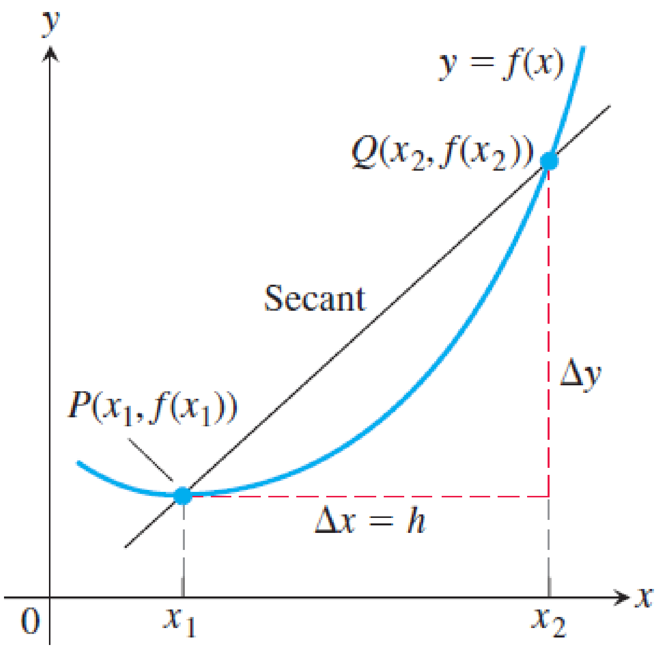


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Secant line vs tangent line



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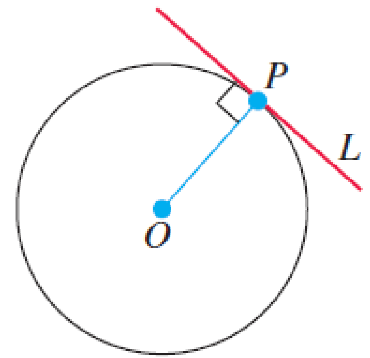

FIGURE 2.2 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

Secant line vs tangent line

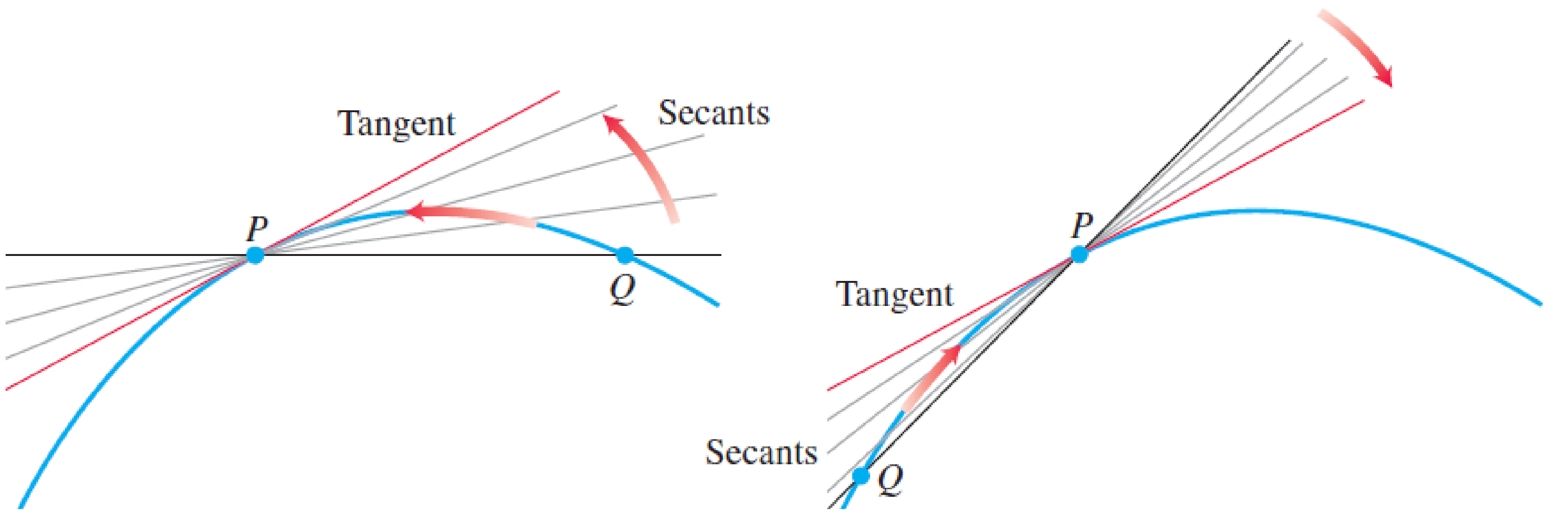

FIGURE 2.3 The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

Instantaneous rate of change and tangent lines ?

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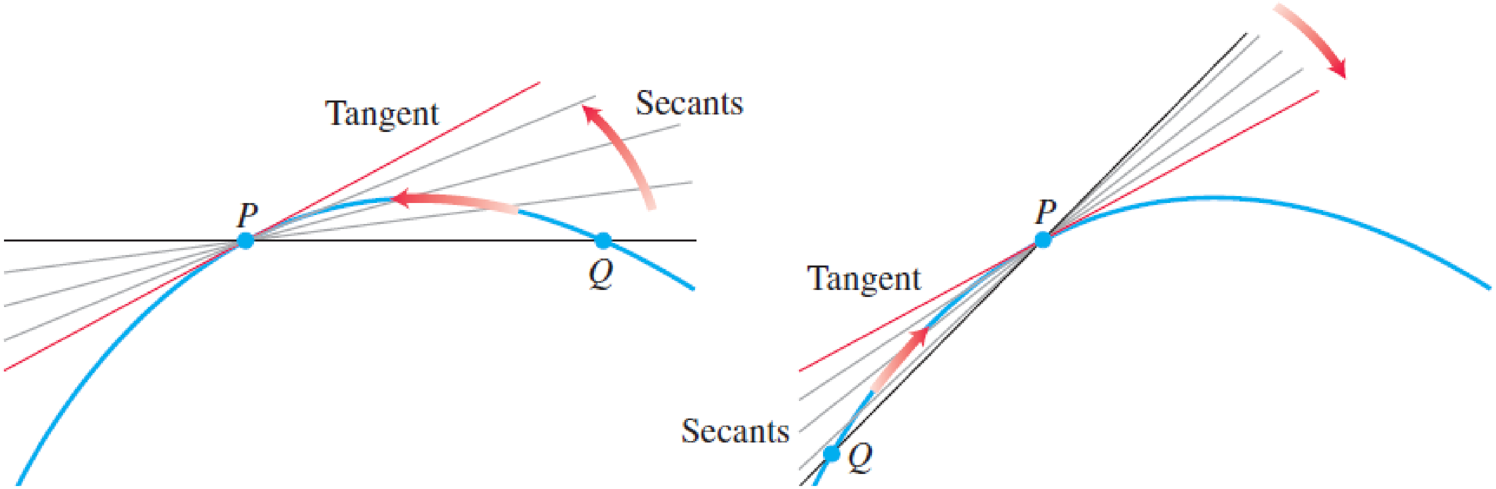

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Limit of function values

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

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TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \frac{x^2-1}{x-1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Limit of a function

Generalizing the idea illustrated in Example 1, suppose $f(x)$ is defined on an open interval about c, *except possibly at c itself*. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, we say that f approaches the **limit** L as x approaches c, and write

$$\lim_{x \to c} f(x) = L,$$

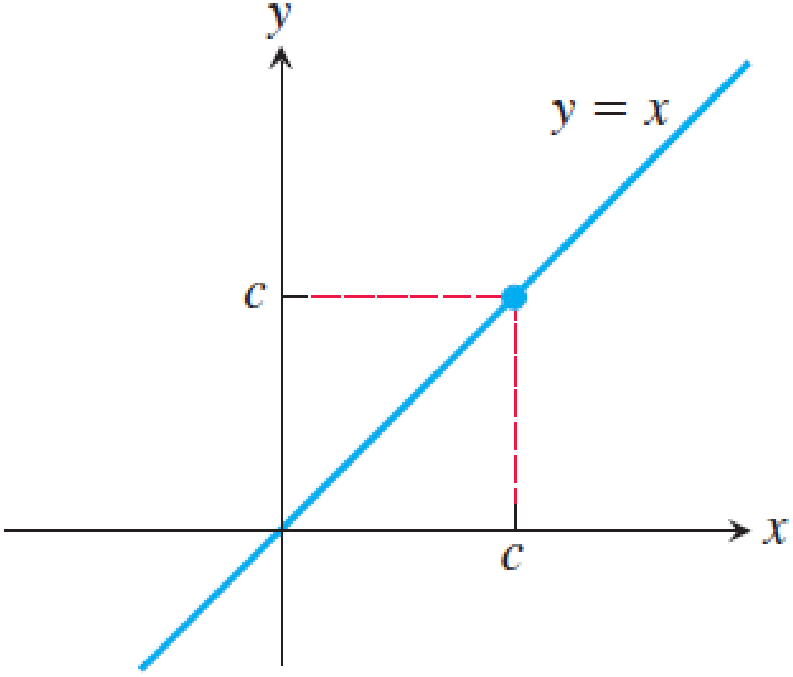
which is read "the limit of $f(x)$ as x approaches c is L." For instance, in Example 1 we would say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Limit of an identity function

If f is the **identity function** $f(x) = x$, then for any value of c (Figure 2.9a),

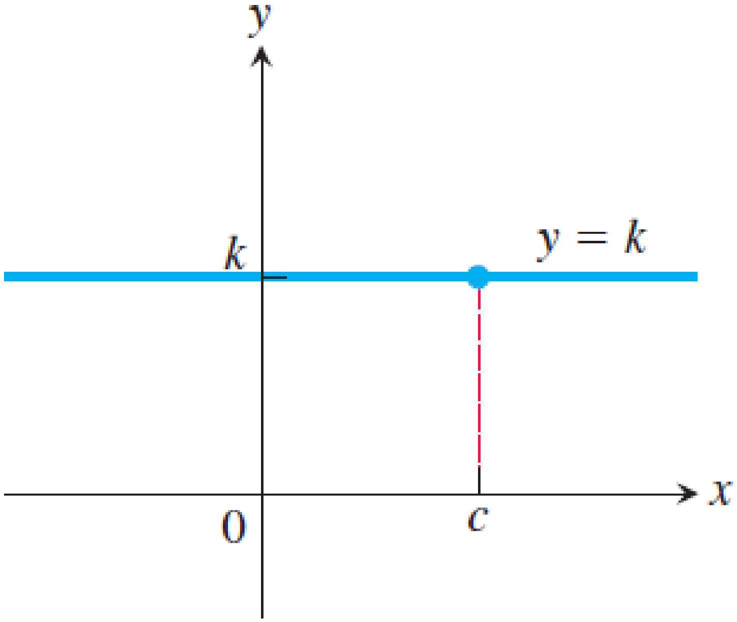
$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c.$$



Limit of a constant function

If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of c (Figure 2.9b),

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k.$$



The limit laws

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

   (If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

Examples

EXAMPLE 5 Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ (Example 3) and the fundamental rules of limits to find the following limits.

(a) $\displaystyle\lim_{x\to c}(x^3 + 4x^2 - 3)$

(b) $\displaystyle\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\displaystyle\lim_{x\to -2}\sqrt{4x^2 - 3}$

Limit of polynomials

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

Exercises:

Find the limits in Exercises 11–22.

11. $\lim_{x \to -3} (x^2 - 13)$

12. $\lim_{x \to 2} (-x^2 + 5x - 2)$

13. $\lim_{t \to 6} 8(t - 5)(t - 7)$

14. $\lim_{x \to -2} (x^3 - 2x^2 + 4x + 8)$

21. $\lim_{h \to 0} \dfrac{3}{\sqrt{3h + 1} + 1}$

Limit of polynomials

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Limit of polynomials

Exercises:

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Limit of rational functions

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Exercises:

23. $\lim_{x \to 5} \frac{x - 5}{x^2 - 25}$

24. $\lim_{x \to -3} \frac{x + 3}{x^2 + 4x + 3}$

25. $\lim_{x \to -5} \frac{x^2 + 3x - 10}{x + 5}$

26. $\lim_{x \to 2} \frac{x^2 - 7x + 10}{x - 2}$

Limit of rational functions

23. $\lim_{x \to 5} \frac{x - 5}{x^2 - 25}$

24. $\lim_{x \to -3} \frac{x + 3}{x^2 + 4x + 3}$

25. $\lim_{x \to -5} \frac{x^2 + 3x - 10}{x + 5}$

26. $\lim_{x \to 2} \frac{x^2 - 7x + 10}{x - 2}$

Eliminating common factor from zero denominators

EXAMPLE 7 Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by Theorem 3:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

One-sided limits: Right-hand limit and Left-hand limit

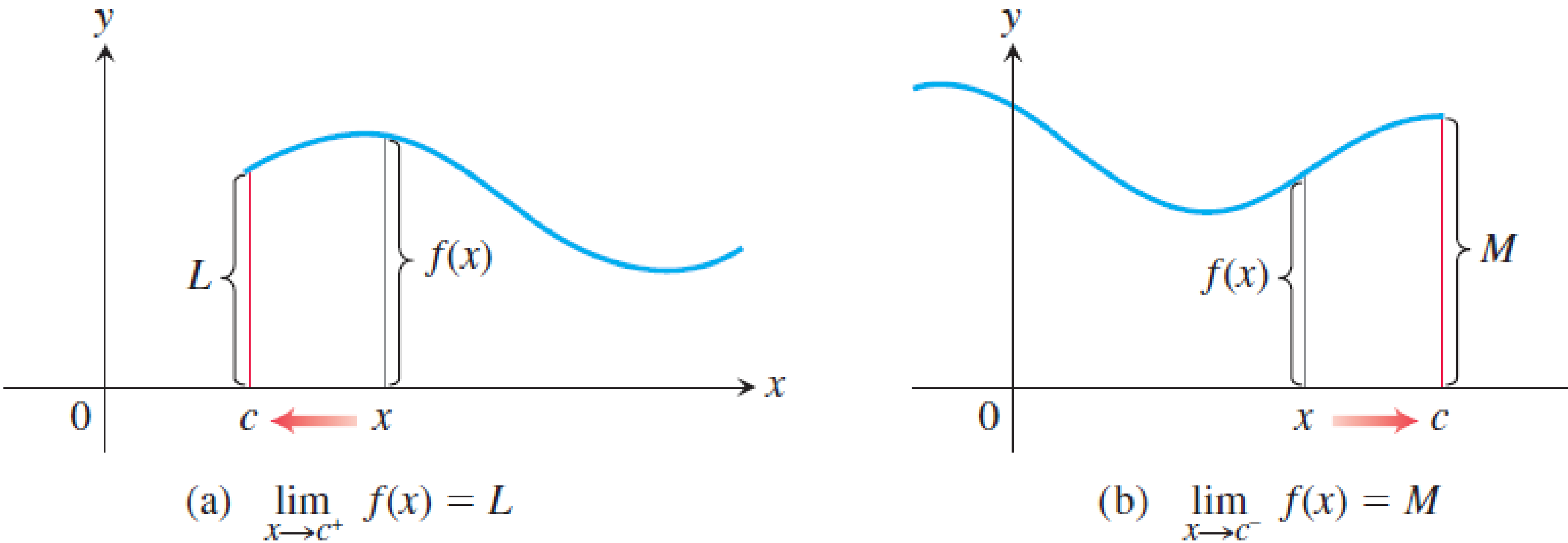

FIGURE 2.25 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

Right-hand limit and left-hand limit

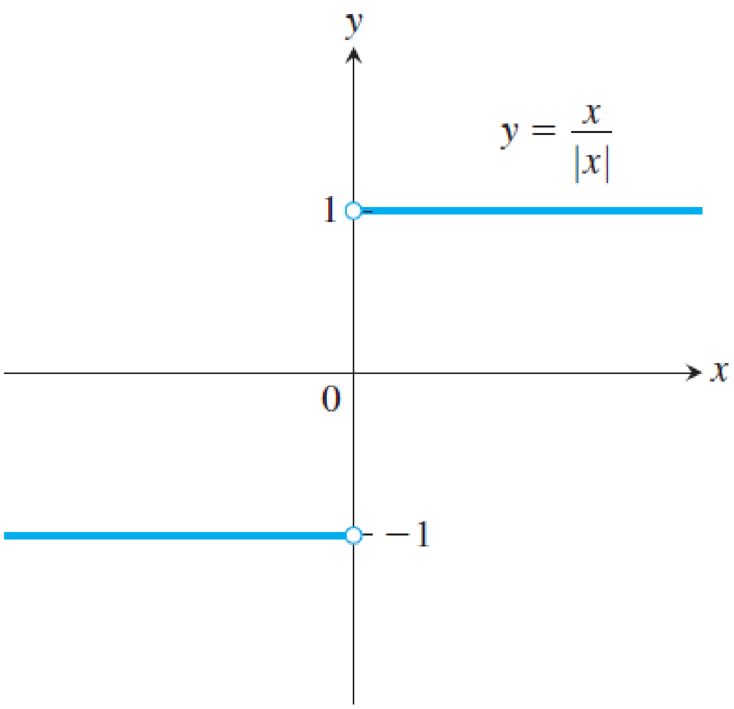

FIGURE 2.24 Different right-hand and left-hand limits at the origin.

$$\lim_{x \to 0^+} f(x) =$$

$$\lim_{x \to 0^-} f(x) =$$